The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and
Doctor of Philosophy Qualifying Exam

Part 1: Classical Physics

August 25, 2004

Work out the solutions to four problems, at least one from each group. Circle the numbers below to indicate your choice of problems.

1  2  3  4  5  6  7
Group A  Group B  Group C

2. Write your code-letter and a page number (in sequential order) on the top right-hand corner of each submitted answer sheet.
3. Write only on one side of the answer sheets.
5. Stack your answer sheets by problem and page number, and then staple them (at the top left-hand corner) with this cover sheet on the top.

Good Luck! (And may you not need it.)
1. A bead of mass $m$, under no external force, is attached by a massless inextensible cord which is completely wound around a cylinder of radius $R$. This cylinder is placed within a concentric cylindrical shell, of radius $3R$. A radially directed kick sends the bead spiraling outward with initial velocity $\vec{v}_0$, unwinding the cord as shown:

\[ \text{[7pts]} \quad a. \text{ Using the length of the unwound piece of the cord, } \ell, \text{ as a generalized coordinate, write down the Lagrangian and determine the equation of motion.} \]

\[ \text{[7pts]} \quad b. \text{ Find the trajectory } \ell = \ell(t) \text{ of the bead.} \]

\[ \text{[7pts]} \quad c. \text{ Find the angular momentum of the bead about the axis of the cylinder and the kinetic energy after a time } t. \]

\[ \text{[4pts]} \quad d. \text{ Find the time when the bead will hit the outer cylinder.} \]
2. A particle of mass $m$ moves under the influence of a potential $V(r) = Kr^4$ where $K > 0$.

[7pts]  
\( a. \) Calculate the force $\vec{F}(r)$ and make plots of both $|\vec{F}(r)|$ and $V(r)$.

[7pts]  
\( b. \) Make a plot of the effective potential and discuss the motion of the particle without solving the equations of motion, for the cases $E < 0$, $E = 0$, and $E > 0$.

[5pts]  
\( c. \) Find the values of total energy $E$, the Lagrangian function $L$ and the radius of a circular orbit.

[3pts]  
\( d. \) Calculate the period of this circular motion.

[3pts]  
\( e. \) Calculate the period of small radial oscillations, that is, the period of the motion when the particle is slightly disturbed from the circular orbit.
3. A ring with mass \( m_1 \) slides over a uniform rod which has a mass \( m_2 \) and length \( \ell \). The rod is pivoted at one end and hangs vertically. The ring is secured to the pivot by a massless spring with the spring constant \( k \) and unstretched length \( r_0 \), and is constrained to slide along the rod without friction. The rod and the ring are set into motion in a vertical plane. The position of the ring and the rod at time \( t \) is given by \( r(t) \) and \( \theta(t) \), as shown in the figure.

\[ \text{[12pts]} \quad a. \text{ Write the Lagrangian for the system.} \]

\[ \text{[5pts]} \quad b. \text{ Obtain the Hamiltonian.} \]

\[ \text{[8pts]} \quad c. \text{ Obtain the differential equation of motion.} \]
4. A grounded conductor has the shape of an infinite horizontal plane, with a hemispherical bulge of radius $R$ (see the figure below). A point-charge $q$ is placed at a distance $h > R$ above the center of the hemisphere.

\[ q \]

\[ h \]

\[ R \]

\[ \text{–4–} \]

[12pts]  a. Using the method of images, determine the total electrostatic potential.

[7pts]  b. Determine the electrostatic force on the original charge.

[6pts]  c. Determine the lowest non-zero term in the multipole expansion of the electrostatic potential.
5. A plane electromagnetic wave is incident on the planar interface between linear, isotropic and homogeneous dielectric media of (real) indices of refraction $n_1$ and $n_2$, at an angle $\theta_i$ from the normal to the interface plane. Assume that the magnetic permeabilities of both dielectrics are $\mu_1, \mu_2 \approx \mu_0$.

[5pts] a. If the incident electric field, $\vec{E}_i$, is parallel to the interface, give the equalities for continuity of both the electric and the magnetic field.

[8pts] b. Derive the ratio of the amplitudes of the reflected and incident electric fields, $r_{TM} = |\vec{E}_{0r}|/|\vec{E}_{0i}|$, as a function of $n_1, n_2$ and the incident and transmitted angles, $\theta_i, \theta_t$.

[6pts] c. In the case when the angle between the direction of the reflected and the transmitted waves is 90°, determine the numerical value of $r_{TM}$.

[6pts] d. In reflection from non-dielectric materials, many of the above assumptions no longer hold; in particular, consider now the case when $n_i = 1$ and $n_2 = n_R + in_I$ is complex. Determine the ratio of the intensities of the reflected and the incident wave, and show that for Gallium ($n_R = 3.7$ and $n_I = 5.4$), $I_r/I_i = 0.7$. 

6. An ideal gas of particles, each of mass $m$, moving in only one dimension and at temperature $T$, is subject to an external force governed by the potential $V(x) = Ax^n$, where $0 \leq x \leq \infty$, and $A, n > 0$.

[12pts] $a$. Calculate the average potential energy per particle.

[7pts] $b$. For $n = 2$ (harmonic oscillator potential), calculate $\langle V \rangle$.

[6pts] $c$. Calculate the average potential energy per particle in a gas in a uniform gravitational field ($n = 1$).
7. The cycle of a highly idealized gasoline engine can be approximated by the so-called Otto cycle (see figure). Treat the working medium as an ideal gas, with $\gamma \overset{\text{def}}{=} C_p/C_v$.

1 → 2: adiabatic compression
3 → 4: adiabatic expansion
2 → 3 and 4 → 1: isohoric

$Q_1$: amount of heat absorbed by the gas
$Q_2$: amount of heat transferred from the gas

[8pts] a. Obtain an expression for the efficiency, $\eta$, of this cycle in terms of the compression ratio $r \overset{\text{def}}{=} V_i/V_f$

[5pts] b. Compute $\eta$ for $\gamma = 1.4$ and $r = 10$.

[7pts] c. Obtain an expression for the work, $W$, done on the gas in the adiabatic compression process 1 → 2 in terms of the initial volume and pressure $V_i, P_i$ and $\gamma$.

[5pts] d. Compute $W$ for $V_i = 2 \text{ L}$ and $P_i = 1 \text{ atm.}$
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Part 2: Modern Physics

August 27, 2004

Work four problems, at least one from each group. Circle the numbers below to indicate your chosen problems.

1  2  3  4  5 6  7
Group A  Group B  Group C

2. Place the code-letter and a page number on the top right-hand corner of each submitted answer sheet.
3. Write only on one side of the answer sheets.
5. Stack your answer sheets by problem and page number, and then staple them (at the top left-hand corner) with this cover sheet on the top.

Good Luck! (And may you not need it.)
1. An electron is described by the wavefunction

\[ \psi(x) = \begin{cases} 
0 & \text{for } x < 0, \\
C e^{-x/x_0} (1 - e^{-x/x_0}) & \text{for } x > 0,
\end{cases} \]

where \( x_0 = 1 \text{ nm} \) and \( C \) is a constant.

[6pts] \( a. \) Calculate the value of \( C \) that normalizes \( \psi(x) \).

[6pts] \( b. \) Where is the electron most likely to be found? That is, calculate the value of \( x \) where the probability of finding the electron is the largest.

[8pts] \( c. \) Calculate the expectation value, \( \langle x \rangle \), for this electron and compare your results with the most likely position. Comment on any differences.

[5pts] \( d. \) Calculate the indeterminacy, \( \Delta x \).
2. Consider a relativistic particle of rest mass $m$, momentum $p$, and total energy $E$.

[8pts]  

a. Show that $E^2 = p^2 + m^2 c^4$.

[8pts]  

b. Show that the “length” of the energy-momentum four-vector of a particle is invariant with respect to all Lorentz transformations.

[9pts]  

c. A proton-proton collision can create a $\pi^0$ meson as an additional particle in the final state, if there is sufficient energy. The observed reaction is $p + p \rightarrow p + p + \pi^0$. If the initial state consists of a proton of kinetic energy $K$ colliding with a proton at rest, calculate the minimum value of $K$ for which the reaction may occur.
3. Charges of $5 \mu C$ are located at points $A$ and $C$, as shown below.

\[ \begin{array}{c}
\text{A} \\
0.1 \text{ m} \\
\text{B} \\
0.1 \text{ m} \\
\text{C} \\
\text{D} \\
1 \text{ m} \\
\end{array} \]

[10pts] a. A bead of 15 g mass and $5 \mu C$ charge is released from rest at point $B$; calculate its speed at point $D$.

[15pts] b. Redo this calculation for a bead of $40 \times 10^{-15}$ g mass; neglect radiative effects.

(Remark: The distance between points $A$ and $C$ is immaterial for answering the questions!)
4. An operator $Q$ satisfies the relations

$$\left[ [Q, \vec{J}^2], \vec{J}^2 \right] = \frac{1}{2} (Q\vec{J}^2 + \vec{J}^2 Q) + \frac{3}{16} Q, \quad [Q, J_z] = m_q Q,$$

where $\vec{J}$ is the usual (total) angular momentum (vector) operator and $J_z$ the component in the $z$ direction.

[6pts] a. For the matrix element $\langle j', m' | Q | j, m \rangle$ to be non-zero, use the first relation to determine the allowed values $\Delta j = j' - j$.

[6pts] b. For the matrix element $\langle j', m' | Q | j, m \rangle$ to be non-zero, use the second relation to determine the allowed values $\Delta m = m' - m$ in terms of $m_q$.

[3pts] c. Given your results for a. and b., what are the two possible values for $m_q$?

[5pts] d. Calculate $\langle j', m' | [Q, \vec{J}^2] | j, m \rangle$ in terms of $\langle j', m' | Q | j, m \rangle$, $j$ and $\Delta j$.

[5pts] e. Writing $Q$ and $Q$ for the two operators corresponding to the two possible values of $m_q$, prove that $QQ$ and $Q\bar{Q}$ commute with $J_z$.

Hint: “Sandwich” the given relations between $\langle j', m' |$ and $| j, m \rangle$. 

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5. Consider a 2-dimensional harmonic oscillator, for which the Hamiltonian can be written as $H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2)$.

[4pts] a. Write down energies of the allowed states of this oscillator (in units of $\hbar\omega$) and specify their degeneracy.

[4pts] b. For a suitably small constant $\alpha$, does a perturbation of the form $V = \alpha x$ change the degeneracy? Why (why not).Dirac? Why (why not)?

[4pts] c. For a suitably small constant $\beta$, does a perturbation of the form $V = \beta x^2$ change the degeneracy? Why (why not)?

[4pts] d. For a suitably small constant $\gamma$, does a perturbation of the form $V = \gamma x^4$ change the degeneracy? Why (why not)?

[4pts] e. Use perturbation theory to calculate the first order shift in the ground state energy, caused by a small perturbation $V = \gamma x^4$.

[5pts] f. For all of the above perturbations and for any arbitrary collection of states, is it necessary to use degenerate perturbation theory? Why (why not)?
6. Consider a particle of mass $M$ constrained to move on a circle of radius $a$ in the $x, y$-plane.

[5pts] a. Write down the Schrödinger equation in terms of the usual cylindrical-polar angle $\phi$.

[5pts] b. Determine the complete set of states, the corresponding energy spectrum and orthonormalize the stationary states.

[5pts] c. Assume now that the particle has charge $q$ and is placed in a small electric field $\vec{E} = \mathcal{E} \hat{e}_x$. Determine the first non-zero perturbative correction to the energy levels.

[5pts] d. Instead of the electric field, apply a small magnetic field $\vec{B} = \mathcal{B} \hat{e}_z$. Determine the first non-zero perturbative correction to the energy levels.

[5pts] e. What is the degeneracy of the unperturbed system (the one with $\mathcal{E} = 0 = \mathcal{B}$)? And with $\mathcal{E} \neq 0 = \mathcal{B}$? And with $\mathcal{E} = 0 \neq \mathcal{B}$?
7. Consider an $L \times L \times L$ cube of metal, wherein the electrons may be treated as if comprised of an ideal gas confined in the cube.

[5pts] a. Write down the wave-function for the electron states and the expression for the energy levels, $E_{\vec{n}}$, where $\vec{n} = (n_x, n_y, n_z)$.

[5pts] b. Let $n = |\vec{n}|$. Determine the number of state between $n$ and $n + dn$. Using the relation between $E_{\vec{n}}$ and $\vec{n}$, and $dE_{\vec{n}}$ and $dn$, eliminate $n$ and $dn$ and obtain the number of states, $dN$, within $[E, E + dE]$.

[5pts] c. Determine the Fermi energy, that is, the energy of the highest occupied state.

[5pts] d. Determine the average kinetic energy of these electrons.

[5pts] e. Determine the pressure of this ideal gas of electrons.