(Fundamental) Physics of Elementary Particles

Covariant derivative & the Christoffel symbol
Spacetime Curvature; Matter–gravity coupling;
Special Solutions (Intro)

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Program

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Coordinate Bases

Basis vectors:

\[ \vec{x}_\mu := (\partial_\mu \vec{r}) \quad \text{and} \quad \vec{x}^\mu := g^{\mu\nu}(x) \vec{x}_\nu, \]

so

\[ A_\mu := \vec{x}_\mu \cdot \vec{A}, \quad A^\mu := \vec{x}^\mu \cdot \vec{A}, \quad \text{and} \quad \vec{A} = A_\mu \vec{x}^\mu = A^\mu \vec{x}_\mu, \]

and

\[ \vec{x}_\mu \cdot \vec{x}_\nu = g_{\mu\nu}(x) \quad \text{and} \quad \vec{x}^\mu \cdot \vec{x}^\nu = g^{\mu\nu}(x). \]

Then

\[ \Gamma^\rho_{\mu\nu} : (\partial_\nu \vec{x}_\mu) = \Gamma^\rho_{\mu\nu} \vec{x}_\rho \quad \text{b/c basis completeness} \]

Straightforwardly,

\[ \Gamma^\rho_{\mu\nu} \vec{x}_\rho := (\partial_\mu \vec{x}_\nu) = (\partial_\mu \partial_\nu \vec{r}) = (\partial_\nu \partial_\mu \vec{r}) = (\partial_\nu \vec{x}_\mu) = \Gamma^\rho_{\nu\mu} \vec{x}_\rho. \]

Also

\[ (\partial_\mu \vec{x}^\rho) = -\Gamma^\rho_{\mu\nu} \vec{x}^\nu \quad \text{b/c} \quad \partial_\mu (\vec{x}_\mu \cdot \vec{x}^\nu = \delta^\nu_\mu) = 0. \]
It then follows:

\[ \bar{A} := A^\rho \bar{x}_\rho \quad \text{&} \quad (\partial_\mu \bar{x}_\nu) =: \Gamma^\rho_{\mu\nu} \bar{x}_\rho \quad \Rightarrow \quad (\partial_\mu \bar{A}) = \left[ (\partial_\mu A^\rho) + \Gamma^\rho_{\mu\nu} A^\nu \right] \bar{x}_\rho; \]

\[ \bar{B} := B_\rho \bar{x}^\rho \quad \text{&} \quad (\partial_\mu \bar{x}^\rho) =: -\Gamma^\rho_{\mu\nu} \bar{x}^\nu \quad \Rightarrow \quad (\partial_\mu \bar{B}) = \left[ (\partial_\mu B_\nu) - \Gamma^\rho_{\mu\nu} B_\rho \right] \bar{x}^\nu. \]

Define:

\[ D_\mu A^\rho := (\partial_\mu A^\rho) + \Gamma^\rho_{\mu\nu} A^\nu \quad \text{and} \quad D_\mu B_\nu := (\partial_\mu B_\nu) - \Gamma^\rho_{\mu\nu} B_\rho. \]

Owing to Weyl’s construction,

\[ T(p, q; w) := C^w \otimes \mathcal{YS} \left[ A \otimes \cdots \otimes A \otimes B \otimes \cdots \otimes B \right] \]

it then follows (product rule) that:

\[
(D_\mu T)_{\rho_1 \cdots \rho_q}^{\nu_1 \cdots \nu_p} = (\partial_\mu T_{\rho_1 \cdots \rho_q}^{\nu_1 \cdots \nu_p}) + \sum_{i=1}^{p} \Gamma^\nu_{\mu \sigma_i} T_{\rho_1 \cdots \sigma_i \cdots \rho_q}^{\nu_1 \cdots \nu_p} - \sum_{i=1}^{q} \Gamma^\nu_{\mu \rho_i} T_{\rho_1 \cdots \sigma_i \cdots \rho_q}^{\nu_1 \cdots \nu_p}. \]
More to the point,
\[ X^{\nu_1 \cdots \nu_p}_{\rho_1 \cdots \rho_q, \mu} := (D_{\mu} T)^{\nu_1 \cdots \nu_p}_{\rho_1 \cdots \rho_q} \]
transforms as a type-(\(p, q+1\)) tensor density of weight \(w\).

And, since a partial derivative doesn't (verify), the \(\Gamma\)-symbol cannot either—so as to compensate:

\[
\Gamma^\rho_{\mu \nu}(x) = \left. \frac{\partial x^\rho}{\partial y^\sigma} \frac{\partial y^\kappa}{\partial x^\mu} \frac{\partial y^\lambda}{\partial x^\nu} \Gamma^\sigma_{\kappa \lambda}(y) \right|_{\{z\}} + \left. \frac{\partial x^\rho}{\partial y^\sigma} \frac{\partial^2 y^\sigma}{\partial x^\mu \partial x^\nu} \right|_{\{z\}}.
\]

is tensorial if and only if the transformation \(x \rightarrow y\) is linear.

In which case, no \(\Gamma_\mu\) is needed in the first place. ☺

True of Cartesian \(\rightarrow\) Cartesian rotations & translations.
Thus, the $\Gamma_\mu$ looks awfully like a gauge potential 4-vector, except for the extra transformation matrix:

$$\Gamma'_\mu = [U]_\mu^\nu U \Gamma_\nu U^{-1} + U \partial_\mu U^{-1}$$

Oh, and one more thing:

$$[A_\mu \cdot \Psi]^\alpha = [A_\mu]_\beta^\alpha \Psi^\beta \quad \leftrightarrow \quad [\Gamma_\mu \cdot V]^\rho = \Gamma_{\mu \nu}^\rho V^\nu.$$  

This is a reflection of the conceptual non-linearity:

- The transformation of phases is spacetime-dependent
- The transformation of spacetime coordinates is spacetime-dependent
- Yang-Mills $A_\mu$ is a spacetime 4-vector of “color”-space matrices.
- The $\Gamma$-symbol is a spacetime 4-vector of spacetime matrices.

Thus, $\mathbb{A}$ looks awfully like a gauge potential 4-vector, except for the extra transformation matrix:
Given the relations

\[(\partial_v \vec{x}_\mu) = \Gamma^\rho_{\mu\nu} \vec{x}_\rho \quad \text{and} \quad \vec{x}_\mu \cdot \vec{x}_v = g_{\mu\nu}(x)\]

a relation between the \(\Gamma\)-symbol and the metric must exist. Indeed,

\[(\partial_\mu g_{v\rho}) = (\partial_\mu (\vec{x}_v \cdot \vec{x}_\rho)) = \Gamma^\sigma_{\mu\nu} \vec{x}_\sigma \cdot \vec{x}_\rho + \vec{x}_v \cdot \Gamma^\sigma_{\mu\rho} \vec{x}_\sigma = g_{\sigma\rho} \Gamma^\sigma_{\mu\nu} + g_{\sigma v} \Gamma^\sigma_{\mu\rho}\]

produces

\[\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} [(\partial_\mu g_{v\sigma}) + (\partial_v g_{\mu\sigma}) - (\partial_\sigma g_{\mu\nu})]\]  

which satisfies

\[D_\mu g_{v\rho} = 0 = D_\mu g^{\nu\rho}. \quad \text{covariantly constant}\]

and vice versa: \(D_\mu g_{v\rho} = 0\) with \(D_\mu = \partial_\mu + \Gamma_\mu\) implies Eq. (\(\heartsuit\)).

This (Christoffel) \(\Gamma\)-symbol is thus \textit{metric}. adj. derived from \(g_{\mu\nu}\).
Spacetime Curvature

The Curvature Tensor

- Just like $F_{\mu\nu} := \frac{he}{igc} [D_\mu, D_\nu]$
- we define

$$R_{\mu\nu\rho}^\sigma := [D_\mu, D_\nu]_{\rho}^\sigma = [(\delta_\lambda^\sigma \partial_v + \Gamma_\lambda^\sigma v^\mu) \Gamma^\lambda_{\mu\rho}] - [(\delta_\lambda^\sigma \partial_\mu + \Gamma_\lambda^\sigma \mu^\rho) \Gamma^\lambda_{\nu\rho}],$$

$$= \partial_\nu \Gamma^\sigma_{\mu\rho} - \partial_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\sigma_{\nu\lambda} \Gamma^\lambda_{\mu\rho} - \Gamma^\sigma_{\mu\lambda} \Gamma^\lambda_{\nu\rho}.$$

- Geometric interpretation:
**Spacetime Curvature**

**Conditions & Contractions**

- Define $R_{\mu\nu\rho\sigma} := R_{\mu\nu\rho}^{\lambda} g_{\lambda\sigma}$ (no such thing for $F_{\mu\nu}$)
- The Riemann tensor satisfies the following identities:
  
  \[
  R_{\mu\nu\rho\sigma} = 0, \quad \text{(non-abelian)} \quad \text{Tr}[F_{\mu\nu}] = 0
  \]
  
  \[
  R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma},
  \]
  
  \[
  R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho},
  \]
  
  \[
  R_{\mu\nu\rho\sigma} = +R_{\rho\sigma\mu\nu},
  \]
  
  \[
  \varepsilon^{\lambda\nu\rho\sigma} R_{\mu\nu\rho\sigma} = 0, \quad \text{1st Bianchi identity}
  \]
  
  \[
  \varepsilon^{\kappa\lambda\mu\nu} D_{\lambda} R_{\mu\nu\rho\sigma} = 0, \quad \text{2nd Bianchi identity}
  \]
- The Riemann tensor is part 1st derivative, part quadratic in $\Gamma_{\mu}$
- ...just as $F_{\mu\nu}$ is part 1st derivative, part quadratic in $A_{\mu}$
- ...of 2nd order in derivatives of the metric, $g_{\mu\nu}$, & homogeneous!
  
  It also involves $g^{\mu\nu}$, which is very non-linear in $g_{\mu\nu}$!
Spacetime Curvature

Conditions & Contractions

• For the Yang-Mills type field strength tensor,

\[ g^{\mu\nu} F_{\mu\nu} \equiv 0, \]  
\[
\begin{align*}
\text{Tr}[F_{\mu\nu}] &= [F_{\mu\nu}]_\alpha^\alpha = 0, \quad \text{for semisimple Lie groups,} \\
\text{Tr}[F_{\mu\nu}] &= F_{\mu\nu}, \quad \text{for } U(1) \text{ factors,}
\end{align*}
\]

• Since all four indices in \( R_{\mu\nu\rho\sigma} \) are of the same type, we can define:

Ricci tensor: \( R_{\mu\rho} := R_{\mu\nu\rho}^\nu, \quad \text{invariant} \)

scalar curvature: \( R := g^{\mu\rho} R_{\mu\rho} = g^{\mu\rho} R_{\mu\nu\rho}^\nu. \)

• It is then possible to define:

• \( S_{\mu\nu\rho\sigma} \), the “pure trace” part, = \( \frac{1}{12} R(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}). \)

• \( E_{\mu\nu\rho\sigma} \), the “semi-traceless” part, = \( (g_{\mu[\rho} S_{\nu]\sigma} - g_{\nu[\rho} S_{\sigma]\mu}); S_{\mu\nu} := R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R. \)

• \( C_{\mu\nu\rho\sigma} \), the fully traceless part, Weyl (conformal curvature) tensor.

• Also: \( \| R_{\mu\nu} \|^2 := R_{\mu\nu} g^{\mu\rho} g^{\nu\sigma} R_{\rho\sigma}, \quad \text{invariant} \)

\( \| R_{\mu\nu\rho}^\sigma \| ^2 := R_{\mu\nu\rho}^\sigma g^{\mu\alpha} g^{\nu\beta} g^{\rho\gamma} g_{\sigma\delta} R_{\alpha\beta\gamma\delta}, \quad \text{invariant} \)
For the Yang-Mills case, the only way to construct a Lagrangian density quadratic in $\mathcal{F}_{\mu\nu}$ is $\propto \text{Tr}[\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}]$.

By the same token, consider:

$$\int \sqrt{-g} \, d^4x \, R_{\mu\nu\rho\sigma} \, g^{\mu\kappa} g^{\nu\lambda} R_{\kappa\lambda\sigma\rho}.$$ 

Varying w.r.t. components of $\mathcal{F}_{\mu\nu}$ produces a 2nd order PDE for $\mathcal{F}_{\mu\nu}$.

Varying w.r.t. components of $g_{\mu\nu}$ produces a 4th order PDE for $g_{\mu\nu}$.

Unlike with Yang-Mills $\mathcal{F}_{\mu\nu}$, we now do have $R$, so:

$$\frac{c^3}{16\pi G_N} \int \sqrt{-g} \, d^4x \, R,$$

is the Einstein-Hilbert action.

So that the units are ML$^2$/T, where $[d^4x] = 4$ and $[g_{\mu\nu}] = 0$.

Varying w.r.t. components of $g_{\mu\nu}$ produces a 2nd order PDE for $g_{\mu\nu}$. 
Covariantizing Lagrangians

Matter–Gravity Coupling

- Varying the Einstein-Hilbert action produces
  \[ G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \]
- This is the 2nd order PDE of motion for \( g_{\mu\nu} \). *Empty spacetime!*
- \( R_{\mu\nu\rho} \) and \( R_{\mu\nu} \) and \( R \) are all (very) nonlinear in \( g_{\mu\nu} \), this is a highly non-trivial, nonlinear PDE system.
- Coupling everything else to this gauge-GCT theory:

\[
S[\phi_i(x)] = \int d^4x \mathcal{L}(\phi_i, (\partial_\mu \phi_i), \cdots; x; C_a)
\]

\[
\rightarrow \int \sqrt{|g|} d^4x \left[ \frac{c^3}{16\pi G_N} R - \mathcal{L}(\phi_i, (D_\mu \phi_i), \cdots; x; C_a) \right]
\]

any and all non-metric/Christoffel fields
Matter–Gravity Coupling

Einstein Equations

- Varying the GCT-covariantized action w.r.t. $g_{\mu \nu}$ produces

**Einstein equations:**

$$R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \frac{8 \pi G_N}{c^4} T_{\mu \nu},$$

where

**Energy-momentum:**

$$T_{\mu \nu} := - \frac{2c}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_M)}{\delta g^{\mu \nu}}$$

- So, the presence of matter curves spacetime.
  - $T_{00}$: energy density
  - $T_{0i} = T_{i0}$: linear momentum density
  - $T_{ik} = T_{ki}$ ($i \neq k$): shear stresses
  - $T_{ii}$ (no sum): normal stresses, called “pressure” if all are equal

$$T^{\mu \nu} := g^{\mu \rho} T_{\rho \sigma} g^{\nu \sigma}$$

$$D_\mu T^{\mu \nu} = 0$$

**continuity equation**

Noether Thm.
Matter–Gravity Coupling

Two Oblique Parallels

By construction,

\[ [A_\mu]_{\alpha^\beta} \leftrightarrow \Gamma_{\mu\nu}^{\alpha}, \text{ not very useful because all indices mix!} \]

\[ [F_{\mu\nu}]_{\alpha^\beta} \leftrightarrow \underline{R}_{\mu\nu\rho}^{\sigma} \]

\[ \vec{E} = (F_{0i}), \vec{B} = (F_{ij}) \quad C_{\mu\nu\rho}^{\sigma}, E_{\mu\nu\rho}^{\sigma}, S_{\mu\nu\rho}^{\sigma} \]

While \((F_{0i})\) and \((F_{ij})\) indeed are irreducible representations of \(SO(0,3) \times G_{YM}\) (i.e., rotations \(\times\) gauge group),

\((R_{0i})\) and \((R_{ij})\) are irreducible representations of neither \(SO(0,3)\) (rotations) nor \(SO(1,3)\) (full Lorentz group).

Although \((A_\mu \leftrightarrow \Gamma_\mu)\) and \((F_{\mu\nu} \leftrightarrow R_{\mu\nu})\) are conceptually analogous, this analogy has technical limitations.

Unh...

Wednesday, February 29, 12
Matter–Gravity Coupling

Two Oblique Parallels

• On the other hand...

• The Einstein equations

\[
\left\{ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} g^{\rho\sigma} \left( \partial_\mu \partial_\rho g_{\nu\sigma} + \partial_\nu \partial_\rho g_{\mu\sigma} \right) + \ldots \right\} = \frac{8\pi G_N}{c^4} T_{\mu\nu}
\]

• remind awfully much of Gauss-Ampère equations

\[
\left\{ \left( \Box A^\mu \right) - \eta^{\mu\nu} \left( \partial_\nu \partial_\rho A^\rho \right) \right\} = \frac{1}{4\pi\varepsilon_0} \frac{4\pi}{c} j^\nu.
\]

• So,

\[ j^\mu_e \longleftrightarrow T_{\mu\nu}, \quad \text{both are Noether currents} \]

\[ A_\mu \longleftrightarrow g_{\mu\nu}, \quad \text{both are “most basic” fields} \]

• Just as every 4-current produces an EM field

• & every EM field specifies the 4-current it needs to support it,

• so are the energy-momentum tensor and spacetime curvature linked and shall not be rendered asunder.
To summarize:

<table>
<thead>
<tr>
<th>EM/YM</th>
<th>GCT</th>
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<tbody>
<tr>
<td></td>
<td>conceptually</td>
</tr>
<tr>
<td>$A_\mu$</td>
<td>$\Gamma_\mu$</td>
</tr>
<tr>
<td>$F_{\mu\nu}$</td>
<td>$R_{\mu\nu}$</td>
</tr>
<tr>
<td>$J_\mu$</td>
<td>?</td>
</tr>
</tbody>
</table>
Special Solutions
( Intro )
A Quick Trick…

Consider the Einstein equations:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G_N}{c^4} T_{\mu\nu}, \]

...the trace of which equates

\[ R - \frac{1}{2} 4R = \frac{8\pi G_N}{c^4} g^{\mu\nu} T_{\mu\nu}, \text{ i.e., } R = - \frac{8\pi G_N}{c^4} g^{\mu\nu} T_{\mu\nu} \]

whereby the Einstein equations are equivalent to

\[ R_{\mu\nu} = \frac{8\pi G_N}{c^4} \left[ T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (g^{\rho\sigma} T_{\rho\sigma}) \right] \]

So,

\[ (R_{\mu\nu} = 0) \iff (T_{\mu\nu} = 0) \]

Ricci-flat spacetimes require/imply no material support

Absence of matter implies/requires Ricci-flat spacetimes

This is not the traceless part of the energy-momentum tensor!

Ricci-flatness
Why is “Ricci-flatness” so important?

Well, construct $R := dx^\mu dx^\nu R_{\mu\nu}$. This is a 2-form.

Taken modulo total derivatives, this defines the 1st Chern class.

Integrals over 2-dimensional submanifolds $X$ are invariants of continuous deformations of $X$, within the spacetime.

More importantly, $R \wedge R = d^4x \, \epsilon_{\mu\nu\rho\sigma} R_{\mu\nu} R_{\rho\sigma}$ is a 4-form.

...and may be integrated over the whole spacetime manifold.

...and is a topological invariant (1st Chern number, $C_1$) of the whole spacetime manifold.

Ricci-flatness implies that $C_1(\text{spacetime}) = 0$. 

I’LL BE BACK.
Special Solutions: Intro

Immaterial (Ricci-flat) Solutions

- Consider empty space.
- That is, space with no matter. (immaterial)
- In 1915, Karl Schwatzschild, while at the Russian front as a German soldier, found the first and best-known Ricci-flat solution to Einstein’s equations. He died within a year.

\[
\begin{align*}
[g_{\mu\nu}] &= \text{diag}(-f_s(r), \frac{1}{f_s(r)}, r^2, r^2 \sin^2(\theta)), \\
 ds^2 &= -f_s(r)c^2 dt^2 + \frac{1}{f_s(r)} dr^2 + r^2 (d\theta^2 + \sin^2(\theta) d\varphi^2), \\
 f_s(r) &= \left(1 - \frac{r_s}{r}\right), \quad r_s = \frac{2G_N M}{c^2}.
\end{align*}
\]

- But, if there was no matter to begin with, whose mass is \(M\)?
- It is the mass of the singularity—a “defect” in spacetime—at the origin.

Empty spacetime can have mass, even classically!
Immaterial (Ricci-flat) Solutions

- Singularity??
  \[ g_{\mu\nu} = \text{diag}(-f_S(r), \frac{1}{f_S(r)}, r^2, r^2 \sin^2(\theta)) \quad f_S(r) := \left(1 - \frac{r_S}{r}\right) \]

- At both \( r = r_S \) and \( r = 0 \), a metric component blows up.
  - At \( r = r_S \), \( f_S(r) = 0 \), the \( dt^2 \)-term vanishes & the \( dr^2 \)-terms blows up.
  - At \( r = 0 \), \( f_S(r) = \infty \), the \( dr^2 \)-term vanishes & the \( dt^2 \)-terms blows up.

- But, that may well be an artifact of “bad” coordinates! Metric components are not invariants; they form a type-(0,2) tensor!

- Indeed, in 1933, Georges Lemaître realized that a coordinate system introduced by Arthur Eddington in 1924 proves that the \( r = r_S \) location is perfectly uneventful.

- In turn, the Kretschmann curvature invariant is
  \[ \| R_{\mu\nu\rho\sigma} \|^2 = \frac{48G_N^2 M^2}{c^4 r^6} \]
Special Solutions: Intro

Immaterial (Ricci-flat) Solutions

- Unh... “the \( r = r_S \) location is perfectly uneventful” is a bit of an understatement.
- Actually, something does happen there:
  \[
  v_1 = \sqrt{\frac{2G_N M}{r}}.
  \]
- is the “escape speed” from a gravitational source of mass \( M \).
- \( r_S = \frac{2G_N M}{c^2} \) \( \Rightarrow \) \( M = \frac{c^2 r_S}{2G_N} \) \( \Rightarrow \) \( v_1 = \sqrt{\frac{2G_N \frac{c^2 r_S}{2G_N}}{r}} = c \sqrt{\frac{r_S}{r}} \)
- ...so the “escape speed” becomes unattainable. Event horizon.
- Oh, and one more thing! Within the event horizon,
  \[
  ds^2 = \left( f_S(r) \right) c^2 dt^2 - \frac{1}{\left| f_S(r) \right|} dr^2 + r^2 (d\theta^2 + \sin^2(\theta) \, d\varphi^2)
  \]
  the \textit{physical meaning of } r \textit{ & } t \textit{ is swapped.}
Special Solutions: Intro

Immaterial (Ricci-flat) Solutions

When discussing Yang-Mills (EM, Strong, Weak) interactions, we assumed a flat, $\mathbb{R}^{1,3}$-like spacetime. Even the “topologically non-trivial” solutions do not change the spacetime. It’s an arena.

In general relativity, non-trivial spacetimes are not $\mathbb{R}^{1,3}$-like.

In so-modeling gravity, we can excise portions of spacetime...though that may render the spacetime somehow incomplete.

Spacetime (non-)singularity may well thus be a subtle issue.

- Geodesically complete: refine: time-like, null, space-like.
- Metrically complete: convergence of all Cauchy sequences.
- B-complete: if every $C^1$-curve of finite length is contained.
- Curvature invariants: $R_{\mu\nu\rho}{}^\sigma$ has 20 independent DoF’s; no known list.

B-completeness implies geodesic completeness, and coincides with metric completeness—only for $g_{\mu\nu} \geq 0$, not for spacetime.
Thanks!

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